

Poisson–Dirichlet Limit Theorems in Combinatorial Applications via Multi-Intensities

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Abstract

We present new, exceptionally efficient proofs of Poisson–Dirichlet limit theorems for the scaled sizes of irreducible components of random elements in the classic combinatorial contexts of arbitrary assemblies, multisets, and selections, when the components generating functions satisfy certain standard hypotheses. The proofs exploit a new criterion for Poisson–Dirichlet limits, originally designed for rapid proofs of Billingsley’s theorem on the scaled sizes of log prime factors of random integers (and some new generalizations).

Unexpectedly, the technique applies in the present combinatorial setting as well, giving, perhaps, a long sought-after unifying point of view. The proofs depend also on formulas of Arratia and Tavaré for the mixed moments of counts of components of various sizes, as well as formulas of Flajolet and Soria for the asymptotics of generating function coefficients.

1 Introduction

1.1 Summary

The goal of this paper is to provide new and exceptionally efficient proofs of very general Poisson–Dirichlet limit theorems for the scaled sizes of components of random elements in the classic combinatorial contexts of assemblies, multisets, and selections, when the components generating functions satisfy certain standard hypotheses. The proofs depend on a fairly new characterization of convergence in distribution to a Poisson–Dirichlet process, originally designed to yield a rapid proof of Billingsley’s 1972 theorem on the asymptotic scaled sizes of log prime factors of random integers. That work, including new generalizations of Billingsley’s result to factorizations in wide classes of normed arithmetic semigroups, was presented in [2].

Poisson–Dirichlet limit theorems are also available for the asymptotic scaled sizes of irreducible components of various random combinatorial objects. The earliest such result was that of Kingman [10] and Vershik and Schmidt [12],

applying to irreducible cycles of random permutations distributed uniformly or, more generally, according to the Ewens sampling formula. Analogous results for other random combinatorial objects were eventually discovered, and by the early 1990's quite general theorems applying uniformly to members of quite general families were known. The first such, due to Jennie Hansen [8], exploited generating function structure. Subsequent versions, due to Arratia, Barbour and Tavaré [1], invoked significantly weaker hypotheses and used much different techniques in combinatorial stochastic processes. Further generalizations continue to be published.

Unexpectedly, the techniques of [2] were found to apply to the combinatorial regime as well, in the presence of generating functions, giving new proofs of results going beyond those of [8] though not as general as those of [1]; but all the classical cases are included, and the new proofs are extremely rapid. The commonality of technique may be viewed as providing a unifying framework for Billingsley's theorem and the combinatorial limit theorems, one which has long been sought. (See e.g. the unpublished [9] by J.F.C. Kingman.¹)

1.2 History, Definitions, Notation

The origins of these limit theorems lie in the following earlier results. Let $p_1 \geq p_2 \geq \dots$ be the prime factors of a random integer chosen uniformly from $[1..n]$, and let

$$L_j := \log p_j / \log n.$$

Or, let $l_1 \geq l_2 \geq \dots$ be the cycle lengths of a uniform random permutation of length n , and let

$$L_j := l_j / n.$$

In either case we have

$$\lim_{n \rightarrow \infty} \Pr(L_1 \leq t) = \rho(1/t)$$

where $\rho(\cdot)$ is Dickman's ρ , the unique continuous function on $[0, \infty]$ satisfying

$$\rho(t) = 1 \text{ for } 0 \leq t \leq 1$$

and

$$t\rho(t) = \int_{t-1}^t \rho(u) du \text{ for } 1 \leq t < \infty, \quad (1)$$

as shown for random integers by Dickman [4] in 1930 and for random permutations by Goncharov [7] in 1944.

Nowadays these can be viewed as respective corollaries of a pair of later results giving the limiting distributions of the the entire joint processes L_1, L_2, \dots ,

¹Kingman proposes one possible unifying vantage point if natural density is replaced by harmonic density, in Billingsley's theorem, but he is apparently dissatisfied with this because recovering the original theorem then seems to require the intervention of quite nontrivial auxiliary results.

in the two cases. To state these results we first define the Poisson–Dirichlet distribution:

Let U_1, U_2, \dots be iid uniform on $[0, 1]$, and define a process G_1, G_2, \dots , also with values lying in $[0, 1]$, via

$$G_1 = 1 - U_1, G_2 = U_1(1 - U_2), G_3 = U_1U_2(1 - U_3), \dots$$

Then the Poisson–Dirichlet process (PD for short) $X_1 \geq X_2 \geq \dots$ is the outcome of sorting G_1, G_2, \dots into non-increasing order, i.e.

$$(X_1 \geq X_2 \geq \dots) = \mathbf{SORT}(G_1, G_2, G_3 \dots).$$

It follows at once from the definition of the G_i ’s that $X_1 + X_2 + \dots = 1$ almost surely, and it can be shown that for each $k > 0$, X_1, \dots, X_k have the marginal distribution with joint density function

$$f_k(x_1, x_2, \dots, x_k) = \frac{1}{x_1 \cdots x_k} \rho\left(\frac{1 - x_1 - \cdots - x_k}{x_k}\right)$$

on $\{1 \geq x_1 \geq x_2 \geq \cdots \geq 0\} \cap \{x_1 + \cdots + x_k \leq 1\}$. (In particular, for $k = 1$ it follows from this, together with (1), that $\Pr(X_1 \leq t) = \rho(1/t)$.) This explicit distribution function provides an alternative characterization of PD. There are a number of other characterizations, though we will not need them here.

More generally, for any real parameter $\theta > 0$ the Poisson–Dirichlet(θ) process (PD(θ)) is defined by replacing each U_i with $U_i^{1/\theta}$ in the definition above. (So in particular, PD(1) is just PD.) Then the density functions f_k are replaced by

$$f_{\theta,k}(x_1, \dots, x_k) = \frac{e^{\gamma\theta} \Gamma(\theta) x_k^{\theta-1}}{x_1 \cdots x_k} g_\theta\left(\frac{1 - x_1 - \cdots - x_k}{x_k}\right)$$

where g_θ is the unique continuous function on $(0, \infty)$ satisfying

$$g_\theta(t) = \frac{e^{-\gamma\theta} t^{\theta-1}}{\Gamma(\theta)} \text{ for } 0 < t \leq 1$$

and

$$tg_\theta(t) = \theta \int_{t-1}^t g_\theta(u) du \text{ for } 1 \leq t.$$

Again there are alternative characterizations which might be more convenient in other contexts; see [1] for full details.

We can now state the two original Poisson–Dirichlet limit theorems:

Theorem 1 (Billingsley, 1972). *Let $p_1 \geq p_2 \geq \dots$ be the prime factors of a uniform random integer $N \in [1..n]$, and define $L_{jn} = \log p_j / \log n$, where the latter sequence is padded out with zeros. Then for each $k > 0$, as $n \rightarrow \infty$ the joint distribution of L_{1n}, \dots, L_{kn} converges weakly to the initial k -dimensional joint PD(1) distribution.*

Theorem 2 (Kingman, 1977; Vershik and Schmidt, 1977). *Let $l_1 \geq l_2 \geq \dots$ be the cycle lengths of a uniform random n -long permutation, and define $L_{jn} = l_j/n$, where the latter sequence is padded out with zeros. Then for each $k > 0$, as $n \rightarrow \infty$ the joint distribution of L_{1n}, \dots, L_{kn} converges weakly to the initial k -dimensional joint PD(1) distribution.*

(Since, as noted, $F(t) = \rho(1/t)$ is the cumulative distribution function of the initial PD variable, the earlier results of Dickman and Goncharev are indeed respective corollaries of the two theorems just stated.)

Billingsley proved his result before PD was a named and studied distribution, deriving his limiting k -dimensional distributions in the form of certain series expansions not obviously involving Dickman's ρ , of which he appears to have been unaware. In turn, neither Kingman, who had both named and made a study of PD(θ) in [10], nor Vershik and Schmidt, makes any mention of Billingsley's theorem

At any rate, as remarked in [9] it was not until 1984 that a publication [11] noted that the two limiting distributions were identical, which immediately raised the question of why this should be so. Over the years, as already mentioned, analogues of the permutation result were proved for other random decomposable objects, with various unified methods of proof. Billingsley's theorem, on the other hand, remained an isolated result in probabilistic number theory until very recently (see [2]); and although several different proofs have appeared, the methods have seemed different from those used for the combinatorial results, and the coincidence of limiting distributions has been felt to lack adequate explanation.

In the next section, however, we will use the recent criterion, from [2], for convergence to PD to give very brief, self-contained proofs of both theorems above by means of a common method; and then we will go on to use the companion characterization of convergence to the more general PD(θ), for the promised combinatorial applications, in Section 3.

2 Characterization via multi-intensities

The following characterizations of convergence to PD and to PD(θ) were originally presented in [2]. In what follows, all (random) multisets are (almost surely) at most countable, with only finite multiplicities.

Given a sequence A_n of random multisubsets of $(0, 1]$, let T_n denote the sum of the elements of A_n , counting multiplicities; and for any multiset A and set S in $(0, 1]$ let $|A \cap S|$ denote the cardinality of the intersection, also counting multiplicities. Also, let $L(n) = (L_1(n), L_2(n), \dots)$, where $L_i(n) :=$ the i^{th} largest element of A_n if $i \leq |A_n|$, and $L_i(n) := 0$ if $i > |A_n|$. (Our hypotheses will ensure that, almost surely, no A_n possesses a positive accumulation point, ensuring in turn that the elements can actually be placed in a non-increasing sequence.)

Here, first, is the PD-only version:

Proposition 1. *Suppose that $T_n \leq 1$ almost surely, for all n , and that for any collection of disjoint closed intervals $I_i = [a_i, b_i] \subset (0, 1]$, $i = 1, \dots, k$ satisfying $b_1 + \dots + b_k < 1$, for any $k \geq 1$, we have*

$$\liminf \mathbb{E} |A_n \cap I_1| \cdots |A_n \cap I_k| \geq \prod_{i=1}^k (\log(b_i) - \log(a_i)) \quad (2)$$

as $n \rightarrow \infty$. Then $L(n)$ converges in distribution to (L_1, L_2, \dots) , the Poisson–Dirichlet distribution with parameter 1.

For arbitrary $\text{PD}(\theta)$ we also have

Proposition 2. *Let $\theta > 0$. Suppose that $T_n \leq 1$ almost surely, for all n , and that for some $-\infty < \alpha, \beta < \infty$ with $\alpha + \beta = 1 - \theta$, it is the case that for any collection of disjoint closed $I_i = [a_i, b_i] \subset (0, 1]$, $i = 1, \dots, k$ satisfying $b_1 + \dots + b_k < 1$, for any $k \geq 1$, we have*

$$\liminf_{n \rightarrow \infty} \mathbb{E} \prod_{i=1}^k |A_n \cap I_i| \geq \frac{\theta^k}{(1 - a_1 - \dots - a_k)^\alpha (1 - b_1 - \dots - b_k)^\beta} \prod_{i=1}^k (\log(b_i) - \log(a_i)). \quad (3)$$

Then $L(n)$ converges in distribution to $(L_1, L_2, \dots)_\theta$, the Poisson–Dirichlet process with parameter θ .

Both propositions are proved in [2]. Note that the condition $T_n \leq 1$ ensures that A_n can possess no positive accumulation point, as promised.

To show at once how rapidly results can be derived using the above criteria, here are complete, self-contained proofs of Theorems 1 and 2, much briefer than any by previous methods. (The present proof of Theorem 1 was already presented in [2], but since it is so brief, and to make the point that there is a single over-arching methodology, we reproduce it here.)

To prove Theorem 1, let A_n be the multiset whose elements are $\log p / \log n$ for all prime factors p of a random $1 \leq N \leq n$, including any multiple copies, and let A_n^1 be the underlying set, i.e. with all positive multiplicities truncated down to 1. For any prime p at all let $I(p|N)$ denote the indicator function of the event $p|N$. Note that since $\log p_1 + \log p_2 + \dots = \log N \leq \log n$ we automatically get $T_n \leq 1$. Note also that for any test interval $[a_i, b_i]$ we have $\log p / \log n \in [a_i, b_i]$ if and only if $n^{a_i} \leq p \leq n^{b_i}$. Thus, writing things out explicitly for $k = 2$, we get

$$\begin{aligned} E\{|A_n \cap [a_1, b_1]| |A_n \cap [a_2, b_2]|\} &\geq E\{|A_n^1 \cap [a_1, b_1]| |A_n^1 \cap [a_2, b_2]|\} \\ &= E \sum_{n^{a_1} \leq p \leq n^{b_1}} I(p|N) \sum_{n^{a_2} \leq q \leq n^{b_2}} I(q|N) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n^{a_1} \leq p \leq n^{b_1}} \sum_{n^{a_2} \leq q \leq n^{b_2}} E\{I(pq|N)\} \\
&= \sum_{n^{a_1} \leq p \leq n^{b_1}} \sum_{n^{a_2} \leq q \leq n^{b_2}} \frac{1}{pq} + O\left(\frac{n^{b_1+b_2}}{n}\right) \\
&= (\log b_1 - \log a_1)(\log b_2 - \log a_2) + o(1).
\end{aligned}$$

The second equality exploits the fact that always $p \neq q$ since they must lie in disjoint intervals, while the third equality depends on the estimate $|\Pr(p|N) - 1/p| \leq 1/n$ together with the fact that there are at most $n^{b_1}n^{b_2}$ summands. The fourth uses Mertens' formula together with the hypothesis that $b_1 + b_2 < 1$. Now take the \liminf as $n \rightarrow \infty$. QED.

To prove Theorem 2, let A_n be the multiset whose elements are the quotients l/n where l ranges over the lengths of all irreducible cycles of a random permutation of length n . Trivially we have $T_n = n$. Note that for any test interval $[a_i, b_i]$ we have $l/n \in [a_i, b_i]$ if and only if $a_i n \leq l \leq b_i n$.

Also, for any positive integer j let C_j be the number of cycles of length j in a random permutation of length n . Then it is well known that provided $j_1 + \dots + j_k \leq n$ we have $E\{C_1 \dots C_k\} = \frac{1}{j_1 \dots j_k}$ exactly. Thus, again writing things out explicitly for $k = 2$, we get

$$\begin{aligned}
&E\{|A_n \cap [a_1, b_1]| |A_n \cap [a_2, b_2]|\} \\
&= E \sum_{a_1 n \leq j \leq b_1 n} C_j \sum_{a_2 n \leq k \leq b_2 n} C_k \\
&= \sum_{a_1 n \leq j \leq b_1 n} \sum_{a_2 n \leq k \leq b_2 n} EC_j C_k \\
&= \sum_{a_1 n \leq j \leq b_1 n} \sum_{a_2 n \leq k \leq b_2 n} \frac{1}{jk} = \left(\sum_{a_1 n \leq j \leq b_1 n} \frac{1}{j} \right) \left(\sum_{a_2 n \leq k \leq b_2 n} \frac{1}{k} \right) \\
&= (\log b_1 - \log a_1)(\log b_2 - \log a_2) + o(1),
\end{aligned}$$

where the requirement $j + k \leq n$ is enforced by the hypothesis $b_1 + b_2 < 1$, and this time we need only know about harmonic sums. Again take the \liminf of both sides, QED.

3 PD limit theorems for general combinatorial families

In this section and the next we present our main results, namely, new proofs of generalizations of the permutation result to randomly selected decomposable combinatorial objects. That is, we prove Poisson–Dirichlet limit theorems for the non-increasing sequence of scaled sizes of irreducible components of random decomposable objects, as total size grows. We cover the three classical families,

namely, Labeled Assemblies, Multisets, and Selections. The objects are chosen equiprobably or, more generally, with their selection probabilities “tilted” to be proportional to

$$\phi^K,$$

where $\phi > 0$ is some fixed parameter and $K = K(\text{object})$ is the total number of irreducible components of an object. (Thus $\phi = 1$ corresponds to equiprobable selection.)

3.1 The Master Theorem

The proofs for the three families will be patterned after the proof of Theorem 2, as presented in Section 2. Specifically, they will all be corollaries of the following Master Theorem. We suppose given a family of objects of various weights or sizes n , where n is a positive integer; that there are finitely many objects of each size n ; that each object decomposes somehow into finitely many irreducible objects, uniquely up to ordering; and that the size of an object is equal to the sum of the sizes of its irreducible components.

Given an object of size n , if K is the total number of irreducible components, let l_1, l_2, \dots, l_K be the sizes of those components, arranged in nonincreasing order. We suppose that any multiple copies are always included so that, e.g., we have $l_1 + l_2 + \dots + l_K = n$. Let C_1, \dots, C_n be the numbers of components of our object, of sizes $1, \dots, n$ respectively; then also $C_1 + 2C_2 + \dots + nC_n = n$. Note that if indices i_1, \dots, i_k have a sum exceeding n , then necessarily at least one of the counts C_{i_1}, \dots, C_{i_k} must vanish.

We are given a sequence of probability distributions, one for each n , on the objects of size n . Thus K , the sizes l_1, l_2, \dots, l_K and the counts C_1, \dots, C_n all become random variables, for each value of n .

Also consider a sequence of families, one family for each n , of k -tuples (i_1, \dots, i_k) of distinct indices $1 \leq i_1, \dots, i_k \leq n$, for fixed k , with all ratios $i_1/n, \dots, i_k/n$ bounded away from 0 as $n \rightarrow \infty$, uniformly over the whole sequence. Call such a sequence a *good sequence of families of k -tuples*.

We can now state the Master Theorem:

Theorem 3. *Suppose our combinatorial objects and probability distributions are such that for some $\theta > 0$ and for any good sequence of families of k -tuples (i_1, \dots, i_k) , the expected values $E\{C_{i_1} \dots C_{i_k}\}$ satisfy*

$$E\{C_{i_1} \dots C_{i_k}\} = \frac{\theta^k}{(1 - m/n)^{1-\theta}} \frac{1}{i_1 i_2 \dots i_k} (1 + o(1)), \quad (4)$$

uniformly over the sequence of families, where we write $m = m(i_1, \dots, i_k) =: i_1 + \dots + i_k$. Then, as $n \rightarrow \infty$ the joint distribution of the initial k -long sequence of scaled sizes

$$l_1/n, \dots, l_k/n$$

converges to the initial k -dimensional projection of $PD(\theta)$.

Proof. We apply Proposition 2. In the notation of that proposition, let A_n be the multisubset containing the elements $l_1/n, l_2/n, \dots$, and let $I_j = [a_j, b_j] \subset (0, 1], j = 1, \dots, k$, be disjoint intervals with $a_j > 0$ for $j = 1, \dots, k$ and with $b_1 + \dots + b_k < 1$. Then we have

$$\begin{aligned} E\{|A_n \cap [a_1, b_1]| \cdots |A_n \cap [a_k, b_k]|\} &= E\left\{ \sum_{a_1 n < i_1 \leq b_1 n} C_{i_1} \cdots \sum_{a_k n < i_k \leq b_k n} C_{i_k} \right\} \\ &= \sum_{a_1 n < i_1 \leq b_1 n} \cdots \sum_{a_k n < i_k \leq b_k n} E\{C_{i_1} \cdots C_{i_k}\} \\ &= \sum_{a_1 n < i_1 \leq b_1 n} \cdots \sum_{a_k n < i_k \leq b_k n} \frac{\theta^k}{(1 - m/n)^{1-\theta}} \frac{1}{i_1 i_2 \cdots i_k} (1 + o(1)) \end{aligned}$$

where we may appeal to (4) in the last step because we claim that the sequence of families of k -tuples of indices arising, as $n \rightarrow \infty$, from the k -fold summations, forms a good sequence: The indices in each k -tuple are distinct because the intervals I_1, \dots, I_k are disjoint, and the ratios $i_1/n, \dots, i_k/n$ are uniformly bounded away from 0 because for each j we have $0 < a_j \leq i_j/n$ where the numbers a_1, \dots, a_k are independent of n .

Note that always, $a_1 + \dots + a_k \leq m/n \leq b_1 + \dots + b_k$. If $\theta \leq 1$, we may then write

$$\begin{aligned} &E\{|A_n \cap [a_1, b_1]| \cdots |A_n \cap [a_k, b_k]|\} \\ &\geq \frac{\theta^k}{(1 - a_1 - \dots - a_k)^{1-\theta}} \left(\sum_{a_1 n < i_1 \leq b_1 n} \frac{1}{i_1} \right) \cdots \left(\sum_{a_k n < i_k \leq b_k n} \frac{1}{i_k} \right) (1 + o(1)) \\ &= \frac{\theta^k}{(1 - a_1 - \dots - a_k)^{1-\theta}} \prod_{j=1}^k (\log b_j - \log a_j) + o(1). \end{aligned}$$

If $\theta > 1$ we proceed in the same way, except that $-a_1 - \dots - a_k$ is replaced with $-b_1 - \dots - b_k$. In either case, take $\liminf_{n \rightarrow \infty}$ of both ends of the inequality and apply Proposition 2, with $\alpha = 1 - \theta$ and $\beta = 0$ or with $\beta = 1 - \theta$ and $\alpha = 0$, respectively. \square

Of course, in any application of Theorem 3, the burden will be the establishment of (4) with the required uniformity guarantees. Conveniently, well-honed tools for this already exist.

3.2 Exp-log asymptotics

We will use two formulas of Flajolet and Soria [6], (5) and (6) below, which we extract from the exposition in [5, Section VII.2], together with several others in the same spirit. They are conveniently packaged consequences of asymptotic formulas of Flajolet and Odlyzko, especially designed for certain combinatorial

applications.² Let $G(z)$ be a function of a complex variable analytic near $z = 0$, whose series expansion at 0 has real non-negative coefficients with finite radius of convergence ρ . We assume, with Flajolet and Soria, that for some $\theta > 0$ and some real λ

FS 1 ρ is the unique singularity of $G(z)$ on $|z| = \rho$;

FS 2 $G(z)$ is continuable to a slightly larger open domain Δ consisting of a disc of radius exceeding ρ centered at 0, but possibly excluding a closed acute-angled wedge domain $|\arg(z - \rho)| \leq \gamma$ with vertex at ρ , for some $0 \leq \gamma < \pi/2$;

FS 3 we have

$$G(z) = \theta \log \frac{1}{1 - z/\rho} + \lambda + O\left(\frac{1}{(\log(1 - z/\rho))^2}\right)$$

as $z \rightarrow \rho$ in Δ .

For later reference, note that

$$\log \frac{1}{1 - z/\rho}$$

itself certainly satisfies all three items, continuing analytically, as it does, to the complement of the real ray $z \geq \rho$.

Given such a G , let $\phi > 0$. Then the formulas of Flajolet and Soria are as follow: the power series coefficients of G around $z = 0$ satisfy

$$[z^n]G(z) = \frac{\theta}{n} \rho^{-n} (1 + O((\log n)^{-2})), \quad (5)$$

and if $F(z) = \exp(\phi G(z))$ then

$$[z^n]F(z) = \frac{e^{\phi\lambda}}{\Gamma(\phi\theta)} n^{\phi\theta-1} \rho^{-n} (1 + O((\log n)^{-2})). \quad (6)$$

Now add the restriction that

$$\rho < 1$$

and assume that the numbers g_i are *integers*. Then if $F(z) = \exp(\phi G(z) + R(z))$, where

$$R(z) := \sum_{j \geq 2} (-1)^{j+1} \phi^j G(z^j)/j \quad (7)$$

then

$$[z^n]F(z) = \frac{C e^{\phi\lambda}}{\Gamma(\phi\theta)} n^{\phi\theta-1} \rho^{-n} (1 + O((\log n)^{-2})) \quad (8)$$

for a certain nonzero constant C to be described.

²Formulas (5) and (6) originally appeared as preliminary results in [6], where they were used as ingredients for various other limit theorems.

Finally, also restrict ϕ to

$$\rho^{-1} > \phi > 0$$

but release the restriction of the numbers g_i to integers. Then if $F(z) = \exp(\phi G(z) + R(z))$, where this time

$$R(z) := \sum_{j \geq 2} \phi^j G(z^j)/j, \quad (9)$$

then we get

$$[z^n]F(z) = \frac{C e^{\phi\lambda}}{\Gamma(\phi\theta)} n^{\phi\theta-1} \rho^{-n} (1 + O((\log n)^{-2})) \quad (10)$$

with $C \neq 0$, same as (8), once again.

As mentioned, (5) and (6) are proved in [5, Section VII.2].³

As for (10), we claim that $R(z)$ as defined in (9) is analytic in an open disc about 0 of some radius exceeding ρ . If so, then since

$$R(z) - R(\rho) = O(z - \rho) = O\left(\frac{1}{(\log(1 - z/\rho))^2}\right)$$

near $z = \rho$, we find that (10) is a corollary of (6), with $C = \exp(R(\rho))$, if $\phi G(z) + R(\rho)/\phi + (R(z) - R(\rho))/\phi$ replaces G in **FS 3**.

To see that $R(z)$ is as claimed, note that for $j \geq 2$ each function $G(z^j)$ is analytic in the open disc of radius $\rho^{1/j} \geq \rho^{1/2} > \rho$, and also that they are uniformly $O(z^2)$ in any closed disc of radius less than $\rho^{1/2}$. Also, when $\phi < \rho^{-1}$, we have $|\phi z| < 1$ in the open disc of radius $\min\{\phi^{-1}, \rho^{1/2}\}$; and this radius exceeds ρ . Therefore, the series defining $R(z)$ converges uniformly and absolutely in any compact subset of that disc. (This argument too was given by Flajolet and Soria, for $\phi = 1$.) This proves (10).

Although the same argument also works for (8), given (7), provided $\phi < \rho^{-1}$, for larger ϕ it gets the series (7) defining $R(z)$ to converge only for $|z| < \phi^{-1} \leq \rho$, which is not good enough. To derive (8) for all positive ϕ we need to look under the hood a bit.

From **FS 3** we have

$$F(z) = \exp(G(z)) = e^\lambda (1 - z/\rho)^{-\theta} \left(1 + O\left(\frac{1}{(\log(1 - z/\rho))^2}\right)\right),$$

and it is *this* formula from which (6) follows via results of Flajolet and Odlyzko; see the discussion in [5, Section VII.2]. From $F(z) = \exp(\phi G(z) + R(z))$, then, we get

$$F(z) = e^{R(z)} e^{\phi\lambda} (1 - z/\rho)^{-\phi\theta} \left(1 + O\left(\frac{1}{(\log(1 - z/\rho))^2}\right)\right). \quad (11)$$

Regardless of the behavior of $R(z)$, we claim that

³Formula (6) is actually proved there for $\phi = 1$; but the more general formula is a trivial corollary of that one.

Lemma 1. *For any fixed $\phi > 0$, $S(z) := e^{R(z)}$ continues analytically to a disc around 0 of radius greater than ρ , and $S(\rho) \neq 0$.*

If so, then from replacing $e^{R(z)}$ in (11) with $S(z) = S(\rho) \times \frac{S(z)}{S(\rho)} = S(\rho) (1 + O(z - \rho))$ near $z = \rho$, we immediately deduce (8), with $C = S(\rho)$. So it remains to prove the lemma, which we now do.

Fix $\phi > 0$. If we restrict to the domain

$$D = \{z : |z| < \min(\phi^{-1}, \rho^{1/2})\},$$

then from rearranging the Taylor expansions of log terms we get

$$R(z) = \sum_{i \geq 1} g_i (\log(1 + \phi z^i) - \phi z^i), \quad (12)$$

a valid identity between analytic functions on D .

Now pick an index $\xi > 0$ for which $\phi \rho^{\xi/2} < 1$, and set

$$T(z) := \sum_{i \geq \xi} g_i (\log(1 + \phi z^i) - \phi z^i).$$

Since on any compact subset of $\{|z| < \rho^{1/2}\}$ the terms $g_i (\log(1 + \phi z^i) - \phi z^i)$ are $O(\phi^2 g_i z^{2i})$, uniformly for $i \geq \xi$, we see from (5) that $T(z)$ defines an analytic function on the open disc $\{|z| < \rho^{1/2}\}$. Also, since we have assumed that the g_i 's are non-negative integers, the expressions $(1 + \phi z^i)^{g_i}$ are polynomials, hence certainly single valued and analytic on the same disc. Therefore the formula

$$S(z) = e^{R(z)} = \left(\prod_{1 \leq i < \xi} ((1 + \phi z^i) \exp(-\phi z^i))^{g_i} \right) e^{T(z)}$$

continues $S(z)$ analytically to the open disc $\{|z| < \rho^{1/2}\}$; and by inspection⁴ we have $S(\rho) \neq 0$. This completes the proof of the lemma and, hence, of (8).

4 The three combinatorial families

4.1 Assemblies

A permutation of length n may be thought of as a partition of $[n] := \{1, \dots, n\}$ into disjoint nonempty blocks, where on each block of size i one of $m_i = (i - 1)!$ possible cycle structures is imposed. More generally, given a sequence m_1, m_2, \dots of positive integers an *assembly of size n* is a partition of $[n]$ into disjoint nonempty blocks, where on each block of size i one of m_i possible structures is imposed, called “irreducible”.⁵ If

$$M(x) = \sum_{i \geq 1} m_i x^i / i!$$

⁴Note that since $S(z)$ does possess zeroes for $|z| < \rho^{1/2}$ when ϕ is large enough, $R(z)$ itself cannot then continue to that domain.

⁵In examples of interest the numbers m_i are not arbitrary, of course – they are the numbers of irreducible combinatorial objects of some sort, of sizes i .

and

$$Q(x) = \sum_{n \geq 0} q(n) x^n / n!$$

are the exponential generating functions for the numbers of irreducible objects of sizes i and the total numbers of assemblies on the set $[n]$, then it is well-known that assemblies are characterized by the formula

$$Q(x) = \exp(M(x)). \quad (13)$$

(Conventionally, we have $q(0) = 1$.) Further, if $q(n, k)$ is the number of objects of size n and with k irreducible components, then if we write

$$q_\phi(n) = \sum_{k=1}^n q(n, k) \phi^k$$

and

$$Q(x, \phi) = \sum_{n \geq 0} q_\phi(n) x^n / n!$$

for some positive parameter ϕ , we have

$$Q(x, \phi) = \exp(\phi M(x)). \quad (14)$$

See, e.g., [3, Section 9.1].

Given a family of assemblies, i.e. given the sequence m_1, m_2, \dots , suppose an assembly of size n is picked at random, either uniformly or, more generally, from the tilted distribution with parameter ϕ . Let C_1, \dots, C_n be the counts of its irreducible components of sizes 1 through n , respectively. For any k -tuple of distinct positive indices i_1, \dots, i_k with $m = i_1 + \dots + i_k \leq n$, the following expression for the mixed moment $E\{C_{i_1} \dots C_{i_k}\}$ is a special case of formula (126) of [3], specialized down to simple products:

$$E\{C_{i_1} \dots C_{i_k}\} = \rho^{-m} \frac{n!}{q_\phi(n)} \frac{q_\phi(n-m)}{(n-m)!} \prod_{j=1}^k \left(\frac{\phi m_{i_j} \rho^{i_j}}{i_j!} \right). \quad (15)$$

We can combine (15) with the Flajolet-Soria formulas discussed above. We suppose we are given a family of assemblies with exponential generating function $M(x) = \sum_{i \geq 1} m_i x^i / i!$ for the numbers of irreducible objects of sizes $1, 2, \dots$.

Lemma 2. *If conditions **FS 1**, **FS 2**, and **FS 3** are satisfied for $G(z) = M(z)$, then for arbitrary $\phi > 0$ we have*

$$E\{C_{i_1} \dots C_{i_k}\} = \frac{(\phi \theta)^k}{(1 - m/n)^{1-\phi \theta} i_1 i_2 \dots i_k} \left(\prod_{j=1}^k \left(1 + O\left(\frac{1}{(\log i_j)^2} \right) \right) \right) + o(1). \quad (16)$$

Proof. With $F(z) = \exp(\phi M(z)) = Q(z, \phi)$, plugging (5) and (6) into (15) immediately yields (16), uniformly over all k -tuples of distinct positive indices i_1, \dots, i_k with $i_1 + \dots + i_k \leq n$. \square

We can now give the main result:

Theorem 4. *Let $l_1 \geq l_2 \geq \dots$ be the irreducible component sizes of a random assembly on the set $[n]$, chosen from a tilted distribution with parameter ϕ , and define $L_{jn} = l_j/n$, where the latter sequence is padded out with zeros. Suppose the Flajolet-Soria conditions **FS 1**, **FS 2**, and **FS 3** are satisfied when $G(z) = M(z)$. Then for each $k > 0$, the joint distribution of L_{1n}, \dots, L_{kn} converges to the initial k -dimensional joint PD($\phi\theta$) distribution.*

Proof. Formula (16) in Lemma 2 looks ready to serve as formula (4) in Theorem 3, except for the k -fold product towards the end of (16). However we are allowed to restrict attention, when applying that theorem, to good families of k -tuples of indices, i.e. with a uniform positive lower bound hypothesis on the ratios $i_1/n, \dots, i_k/n$. This converts the k -fold product to $(1 + o(1))$. Now apply the theorem. \square

4.2 Multisets and Selections

Multisets and Selections are sufficiently alike that we can treat them simultaneously, in parallel.

A monic polynomial of degree n , over some finite field, may be unambiguously identified with the multiset consisting of its irreducible monic factors – “multiset”, because some factors could appear with multiplicities; and the degrees add up to n . Or, if we are interested only in squarefree polynomials, then the irreducible factors form a set, with no repetition of elements; and the degrees still add up to n . More generally, suppose we are given some universe of “irreducible” objects having positive integer weights, with exactly m_i different kinds of irreducibles of weight i . Our two polynomial examples are prototypical of the following two respective constructions.

- A *combinatorial multiset of weight n* is a multisubset of our universe, with total weight n . Equivalently, the *integer* n is partitioned into positive summands, and for each summand i one of the m_i possible summands of weight i is selected, with replacement.
- A *combinatorial selection of weight n* is a subset of our universe, of total weight n . So all components of an object must be of distinct kind, though distinct components of the same weights are permitted.

So the selection construction could be viewed as a subclass of the multiset construction. Note that any additional structure associated with our universe, for instance the fact that a collection of irreducible polynomials multiplies together to form another polynomial, need not be considered in discussion of counting formulas.

For either construction, let

$$M(x) = \sum_{i \geq 1} m_i x^i$$

be the ordinary generating function for the numbers of irreducibles of weight i . Also, if $q(n, k)$ denotes the number of multisets of total weight n containing k irreducibles, including multiplicities, or if we let it denote the number of selections of total weight n containing k irreducibles, then in either case, for any given positive ϕ write

$$q_\phi(n) = \sum_{k=1}^n q(n, k) \phi^k$$

and

$$Q(x, \phi) = \sum_{n \geq 0} q_\phi(n) x^n.$$

For $\phi = 1$, in either case, the series $Q = Q(x, 1)$ reduces to the ordinary generating function for the numbers of composite objects of weights n .

The following two formulas connecting $Q(x, \phi)$ and $M(x)$ are well-known: For multisets we have

$$Q(x, \phi) = \prod_{i \geq 1} (1 - \phi x^i)^{-m_i} = \exp \left(\sum_{j \geq 1} \phi^j M(x^j) / j \right), \quad (17)$$

and for selections we have

$$Q(x, \phi) = \prod_{i \geq 1} (1 + \phi x^i)^{m_i} = \exp \left(\sum_{j \geq 1} (-1)^{j+1} \phi^j M(x^j) / j \right). \quad (18)$$

See, e.g., [3, Section 9.2] for 17 and Section 9.3 for 18.

In either construction, given the set of all composite objects of total weight n constructed from some given universe of irreducibles, suppose one object is picked at random according to the tilted distribution with tilting parameter ϕ . Let C_1, \dots, C_n be the numbers of irreducible components of that object, of weights 1 through n respectively, including any multiple occurrences. (So again $C_1 + 2C_2 + \dots + nC_n = n$.)

For any sequence of positive indices i_1, \dots, i_k with $i_1 + \dots + i_k \leq n$ we have a formula for the corresponding mixed moment. For the multiset construction it is

$$E\{C_{i_1} \dots C_{i_k}\} = \frac{m_{i_1} \dots m_{i_k}}{q_\phi(n)} \sum_{h_1, \dots, h_k \geq 1} \phi^{h_1 + \dots + h_k} q_\phi(n - h_1 i_1 - \dots - h_k i_k), \quad (19)$$

and for the selection construction it is

$$E\{C_{i_1} \dots C_{i_k}\} = \frac{m_{i_1} \dots m_{i_k}}{q_\phi(n)} \sum_{h_1, \dots, h_k \geq 1} (-1)^{h_1 + \dots + h_k + k} \phi^{h_1 + \dots + h_k} q_\phi(n - h_1 i_1 - \dots - h_k i_k). \quad (20)$$

(See formulas (139) and (146) in [3, Sections 9.2 and 9.3], respectively. While the authors give explicit formulas only for the individual falling factorial moments, their method of proof easily yields the present formulas as well.) Note that because of the expressions $q_\phi(n - h_1 i_1 - \dots - h_k i_k)$, the sums have finitely many terms. In our application to Theorem 5 below only the leading term in each case, where $h_1 = \dots = h_k = 1$, will matter asymptotically.

We can marry the Flajolet-Soria asymptotics to the moment formulas (19) and (20):

Lemma 3. *Suppose that conditions **FS 1**, **FS 2**, and **FS 3** are satisfied for $G(z) = M(z)$, the ordinary generating function of the m_i 's, and that for the multiset construction we impose $\phi < \rho^{-1}$. For the selection construction we allow ϕ to be arbitrarily large. Then in either case we have*

$$E\{C_{i_1} \dots C_{i_k}\} = \frac{(\phi\theta)^k}{(1 - m/n)^{1-\phi\theta}} \frac{1}{i_1 i_2 \dots i_k} \left(1 + O(\lfloor n/i_1 \rfloor \dots \lfloor n/i_k \rfloor \rho^{\min(i_1, \dots, i_k)})\right), \quad (21)$$

where $m = i_1 + \dots + i_k$.

Proof. Note that in the present cases the radius of convergence ρ of $G(z) = M(z)$ must satisfy $\rho < 1$, for the trivial reason that otherwise the coefficients of $G(z)$ as given in (5) could not yield integers, for large enough n .

That being so, substitute (5) and either (10) or (8) into (19) or (20) respectively. It is then straightforward to get (21). \square

We can now give the Poisson-Dirichlet limit theorem for multisets and selections. We suppose we are given a universe of irreducibles with ordinary generating function $M(x)$ for the numbers m_i of different kinds of weight i , for $i \geq 1$. Let $\rho < 1$ be the radius of convergence of M .

Theorem 5. *Let $l_1 \geq l_2 \geq \dots$ be the irreducible component sizes of a random multiset or a random selection of weight n , chosen from a tilted distribution with parameter ϕ , where for multisets we suppose that $\phi < 1/\rho$. Define $L_{jn} = l_j/n$, where the latter sequence is padded out with zeros. Suppose the Flajolet-Soria conditions **FS 1**, **FS 2**, and **FS 3** are satisfied when $G(z) = M(z)$. Then for each $k > 0$, the joint distribution of L_{1n}, \dots, L_{kn} converges to the initial k -dimensional joint $PD(\phi\theta)$ distribution.*

Proof. We appeal once again to Theorem 3. Restricting to good sequences of families of k -tuples, together with the fact that $\rho < 1$, converts the multiplicative error term in (21) to $1 + o(1)$, as required in (4). So the theorem applies. \square

Remark. The necessity for the restriction to $\phi < \rho^{-1}$ for multisets may appear to be an artifact of our complex analytic methodology, but the same restriction is also imposed with the methodology of [1]. In fact, as far as we are aware, the limiting behavior for $\phi \geq \rho^{-1}$ is unknown, for multisets.

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